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NOTE

ON LIMITS OF FINITE GRAPHS

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We prove that for any weakly convergent sequence of finite graphs with bounded vertex degrees, there exists a topological limit graphing.

1. Introduction

First let us recall the notion of weak convergence of finite graphs [2]. We need some definitions and notations. A *rooted graph* is a simple graph with a distinguished vertex (the root). Two rooted graphs are called rooted isomorphic if there exists a graph isomorphism between them mapping one root to the other one. A rooted r-ball is a rooted graph $G(V,E)$ such that $\sup_{y\in V(G)} d_G(x,y)\leq r$, where x is the root of G and d_G is the shortest path distance. For $d\geq 1$, we denote by $\mathcal{U}^{r,d}$ the finite set of rooted isomorphism classes of rooted r-balls with vertex degrees bounded by d. If $G(V,E)$ is a graph with $\sup_{y\in V(G)} \deg(y)\leq d$ and $\mathcal{A}\in\mathcal{U}^{r,d}$, then $T(G,\mathcal{A})$ denotes the set of vertices v in $V(G)$ such that $\mathcal{A}$ represents the rooted isomorphism class of the r-neighborhood of v, $B_r(v)$. Set $p_G(\mathcal{A}):=\frac{|T(G,\mathcal{A})|}{|V(G)|}$.

That is G determines a probability distribution on $\mathcal{U}^{r,d}$ for any $r\geq 1$. Now let $\{G_n\}_{n=1}^\infty$ be a sequence of finite simple connected graphs with vertex degrees bounded by d and $\lim_{n\to\infty}|V(G_n)|=\infty$. We say that $\{G_n\}_{n=1}^\infty$ is *weakly convergent* if for any $r\geq 1$ and $\mathcal{A}\in\mathcal{U}^{r,d}$, $\lim_{n\to\infty} p_{G_n}(\mathcal{A})$ exists.

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Now let X be a standard Borel space with a measure μ. Suppose that $T_1, T_2, \dots, T_k$ are measure preserving Borel involutions of X. Then the system $\mathcal{G} = \{X, T_1, T_2, \dots, T_k, \mu\}$ is called a *measurable graphing*. A measurable graphing $\mathcal{G}$ determines an equivalence relation on the points of X. Simply, $x \sim_{\mathcal{G}} y$ if there exists a sequence of points $\{x_1, x_2, \dots, x_m\} \subset X$ such that

- $x_1 = x, x_m = y$
- $x_{i+1} = T_j(x_i)$ for some $1 \le j \le k$.

Thus there exist natural simple graph structures on the equivalence classes, the leafgraphs. Here x is adjacent to y, if $x \neq y$ and $T_j(x) = y$ for some $1 \le j \le k$. Now if $\mathcal{A} \in \mathcal{U}^{r,d}$, we denote by $p_{\mathcal{G}}(\mathcal{A})$ the μ-measure of the points x in X such that the rooted r-neighborhood of x in $\mathcal{A}$ in its leafgraph is isomorphic to $\mathcal{A}$. We say that $\mathcal{G}$ is the limit graphing of the weakly convergent sequence $\{G_n\}_{n=1}^{\infty}$, if for any $r \ge 1$ and $\mathcal{A} \in \mathcal{U}^{r,d}$

$$\lim_{n \to \infty} p_{G_n}(\mathcal{A}) = p_{\mathcal{G}}(\mathcal{A})\,.$$

Combining the ideas of [2] and [3] one can see that for any weakly convergent graph sequence there exists a measurable limit graphing $\mathcal{G}$. Note that for dense graph sequences an analogous construction is given in [4].

If X is a compact metric space with a Borel measure μ and $T_1, T_2, \dots, T_k$ are continuous measure preserving involutions of X, then $\mathcal{G} = \{X, T_1, T_2, \dots, T_k, \mu\}$ is a *topological graphing*. The goal of this note is to give a simple, self-contained proof of the following theorem.

Theorem 1. *If $\{G_n\}_{n=1}^{\infty}$ is a weakly convergent system of finite connected graphs, then there exists a topological graphing $\mathcal{G}$ such that for any $r \ge 1$ and $\mathcal{A} \in \mathcal{U}^{r,d}$, $\lim_{n \to \infty} p_{G_n}(\mathcal{A}) = p_{\mathcal{G}}(\mathcal{A})$.*

2. The construction of the limit graphing

Our graphing construction is motivated by the ideas of [2], [1] and [3]. Let $\{G_n\}_{n=1}^{\infty}$ be a weakly convergent sequence of finite connected graphs, such that $|V(G_n)| \to \infty$ and $\sup_{v \in V(G_n)} \deg(y) \le d$ for all $n \ge 1$. Let S be a set of $d+1$ elements. Color the edges of each G_n by the elements of S to form an edge-coloring, that is if two edges have a joint vertex then they have different colors. This can be done by Vizing's Theorem. Now pick a sequence of integers $r_1 \le r_2 \le \dots$ with the following property: Any graph G with vertex degree bound d has a vertex coloring by r_i colors such that if $d_G(x,y) \le i$, then x and y are colored differently. Obviously if $r_i > (d+1)^i$, then these conditions are satisfied. Now for any $i \ge 1$ and $n \ge 1$ we fix a vertex coloring of G_n by the elements of a set Q_i, $|Q_i|=r_i$, satisfying the coloring conditions above, that is if $d_G(x,y)\leq i$, then x and y are colored differently. We need a refined version of rooted graph isomorphism. Let G and H be two rooted r-balls with vertex degree bound d equipped with an edge-coloring by S and a vertex-coloring with $\prod_{i=1}^r Q_i$ (satisfying the appropriate coloring conditions for each $i\geq 1$). Then G and H are called colored-isomorphic if there exists a graph isomorphism $\phi:G\to H$, mapping root to root and preserving both the edge and the vertex colorings. Let $\mathcal{V}^{r,d}$ be the finite set of colored isomorphism classes.

Again, for each G_n and $r\geq 1$ one obtains a probability distribution on $\mathcal{V}^{r,d}$. Let $p^c_{G_n}(\mathcal{A})$ be the probability of a colored type $\mathcal{A}\in\mathcal{V}^{r,d}$. We may suppose that for any $\mathcal{A}\in\mathcal{V}^{r,d}$, $\lim_{n\to\infty}p^c_{G_n}(\mathcal{A})$ exists (otherwise we can pick an appropriate subsequence). Let $\mathcal{W}^{r,d}_G\subseteq\mathcal{V}^{r,d}$ be the set of colored types $\mathcal{A}$ such that $\lim_{n\to\infty}p^c_{G_n}(\mathcal{A})\neq 0$. Let $\mathcal{A}_{r-1}\in\mathcal{W}^{r-1,d}_G, \mathcal{A}_r\in\mathcal{W}^{r,d}_G$, then let $\mathcal{A}_{r-1}\prec\mathcal{A}_r$ if the $r-1$-ball around the root of $\mathcal{A}_r$ is just $\mathcal{A}_{r-1}$. Since the vertices of $\mathcal{A}_r$ are colored by $\prod_{i=1}^r Q_i$ we can consider the restricted $\prod_{i=1}^{r-1}Q_i$-coloring on the $r-1$-ball around the root, thus the definition of $\prec$ is meaningful. We call an infinite sequence of classes $\mathcal{A}_1\prec\mathcal{A}_2\prec\mathcal{A}_3\prec\dots$ a chain. As usual the distance of two chains $\{\mathcal{A}_i\}_{i=1}^\infty$ and $\{\mathcal{B}_i\}_{i=1}^\infty$ is defined as 2^{-r}, where r is the minimal index for which $\mathcal{A}_r\neq\mathcal{B}_r$. Denote the space of chains by X. X is clearly a compact metric space. If $a\in S$, then $T_a:X\to X$ is defined the following way. Let $x=\{\mathcal{A}_1\prec\mathcal{A}_2\prec\dots\}\in X$. If there is no edge colored by a, incident to the root of x, then let $T_a(x)=x$. If there exists such an edge, then the root of $\mathcal{A}_r$ is connected with this a-colored edge to a unique vertex p of $\mathcal{A}_r$ with $r-1$-ball $\mathcal{B}_{r-1}$. Clearly we have a chain $y=\{\mathcal{B}_1\prec\mathcal{B}_2\prec\dots\}\in X$. Define $T_a(x)=y$. Then T_a is a continuous involution.

Now we define the measure μ. Let $M(\mathcal{A}_r)$ be the open-closed set of chains starting with a sequence $\mathcal{A}_1\prec\mathcal{A}_2\prec\cdots\prec\mathcal{A}_r$, $\mathcal{A}_r\in\mathcal{W}^{r,d}_G$. Then let $\mu(M(\mathcal{A}_r))=\lim_{n\to\infty}p^c_{G_n}(\mathcal{A}_r)$. This way we define a Borel probability measure on X.

Proposition 2.1. *μ is an invariant measure.*

Proof. It is enough to prove, that if M is an open-closed set as above and $a\in S$, then $\mu(M)=\mu(T_a(M))$. Again, if $\mathcal{A}_r\in\mathcal{W}^{r,d}_G$, then $\tau(G_n,\mathcal{A}_r)$ denotes the number of vertices x of G_n such that the r-ball around x has colored type $\mathcal{A}_r$. Note that

$$T_a(M(\mathcal{A}_r))=\dot{\bigcup_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}}M(\mathcal{B}_{r+1})\,, \tag{1}$$

where $\mathcal{B}_{r+1}\,|\,\mathcal{A}_r\sim a$ means that the summation is taken over such $\mathcal{B}_{r+1}$ that

- either there is no edge colored by a going out from the root x of $\mathcal{B}_{r+1}$ and the r-ball around x is of type $\mathcal{A}_r$
- or there exists $(x,y)\in E(G_n)$ colored by a and the r-ball around y is of type $\mathcal{A}_r$.

Also note that

$$\tau(G_n,\mathcal{A}_r)=\sum_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}\tau(G_n,\mathcal{B}_{r+1})\,. \tag{2}$$

Then

$$\begin{aligned}\mu(T_a(M(\mathcal{A}_r)))&=\sum_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}\mu(M(\mathcal{B}_{r+1}))=\\
&=\sum_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}\lim_{n\to\infty}p^c_{G_n}(\mathcal{B}_{r+1})=\lim_{n\to\infty}\sum_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}p^c_{G_n}(\mathcal{B}_{r+1})=\\
&=\lim_{n\to\infty}\sum_{\mathcal{B}_{r+1}|\,\mathcal{A}_r\sim a}\frac{\tau(G_n,\mathcal{B}_{r+1})}{|V(G_n)|}=\lim_{n\to\infty}\frac{\tau(G_n,\mathcal{A}_r)}{|V(G_n)|}=\\
&=\lim_{n\to\infty}p^c_{G_n}(\mathcal{A}_r)=\mu(M(\mathcal{A}_r))\,.\end{aligned}$$

Hence the proposition follows. ∎

In order to finish the proof of Theorem 1 we only need to prove the following proposition.

Proposition 2.2. *If $x=\{\mathcal{A}_1\prec\mathcal{A}_2\prec\ldots\}\in X$ then the r-ball around x in the leafgraph of x has the same rooted isomorphism class as $\mathcal{A}_r$.*

The proof shall be given in two lemmas.

Lemma 2.1. *Let $x=\{\mathcal{A}_1\prec\mathcal{A}_2\prec\ldots\}\in X$ be a chain. Let $\{i_1,i_2,\ldots,i_s\}$ and $\{j_1,j_2,\ldots,j_t\}$ sequences of elements of S such that $s,t\leq r$. Suppose that $T_{i_1}T_{i_2}\cdots T_{i_s}(v)=T_{j_1}T_{j_2}\cdots T_{j_t}(v)$, where v is the root of $\mathcal{A}_r$. Then $T_{i_1}T_{i_2}\cdots T_{i_s}(x)=T_{j_1}T_{j_2}\cdots T_{j_t}(x)$. Note that the involutions are defined both on the graphs and on the space X.*

Proof. For any $r'>r$, if $\mathcal{A}_r\prec\mathcal{A}_{r'}$, then $T_{i_1}T_{i_2}\cdots T_{i_s}(v)=T_{j_1}T_{j_2}\cdots T_{j_t}(v)$ holds for $\mathcal{A}_{r'}$. Thus $T_{i_1}T_{i_2}\cdots T_{i_s}(x)=T_{j_1}T_{j_2}\cdots T_{j_t}(x)$. ∎

Lemma 2.2. *Let x be as above and $T_{i_1}T_{i_2}\cdots T_{i_s}(v)\neq T_{j_1}T_{j_2}\cdots T_{j_t}(v)$. Then $T_{i_1}T_{i_2}\cdots T_{i_s}(x)\neq T_{j_1}T_{j_2}\cdots T_{j_t}(x)$.*

Proof. This is the point where we need the vertex coloring. Recall that in $\mathcal{A}_{2r}$, $T_{i_1}T_{i_2}\cdots T_{i_s}(v)$ and $T_{j_1}T_{j_2}\cdots T_{j_t}(v)$ are colored by $\prod_{i=1}^{2r} Q_i$ and their Q_{2r}-colors are different. Hence if

$$T_{i_1}T_{i_2}\cdots T_{i_s}(x) = \{\mathcal{B}_1 \prec \mathcal{B}_2 \prec \cdots \prec \mathcal{B}_{2r} \prec \dots\}$$

and

$$T_{j_1}T_{j_2}\cdots T_{j_t}(x) = \{\mathcal{C}_1 \prec \mathcal{C}_2 \prec \cdots \prec \mathcal{C}_{2r} \prec \dots\}$$

then $\mathcal{B}_{2r} \neq \mathcal{C}_{2r}$. Therefore $T_{i_1}T_{i_2}\cdots T_{i_s}(x) \neq T_{j_1}T_{j_2}\cdots T_{j_t}(x)$. ∎

Again let $x = \{\mathcal{A}_1 \prec \mathcal{A}_2 \prec \dots\}$ and v be the root of $\mathcal{A}_r$. Suppose that $w \in \mathcal{A}_r$ and $w = T_{i_1}T_{i_2}\cdots T_{i_s}(v)$. Then let $\phi(w) := T_{i_1}T_{i_2}\cdots T_{i_s}(x)$. By Lemma 2.1, ϕ is well-defined and by Lemma 2.2, ϕ is injective. Since any element of the r-ball around x in the leaf-graph of x is in the form $T_{j_1}T_{j_2}\cdots T_{j_t}(x)$, for some $t \leq r$ and $\{j_1, j_2, \dots, j_t\}$, ϕ is also surjective. Hence ϕ is a bijection between $\mathcal{A}_r$ and the r-ball around x in the leaf graph of x. Note that ϕ preserves adjacency. Thus our proposition and consequently Theorem 1 follow. ∎

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